On the different parametrizations of the Q-exponential family distribution

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1 q-exponential family

The density (Eq. (18) of Naudt (2007)) is defined as

$$f_\theta(x) = c(x)\exp_q(-\alpha(\theta) - \theta H(x)),$$

where c, α and H are known functions. Furthermore, $\exp_q$ is the q-deformed exponential function defined as

$$\exp_q(z) = [1 + (1-q)z]_+^{1/(1-q)} \text{ for } z \in \mathbb{R}, q \neq 1,$$

where $[z]_+ = \max(z, 0)$. $\exp_q$ is construct as the inverse of the q-deformed logarithm defined as

$$\log_q(z) = \frac{z^{1-q} - 1}{1-q} \text{ for } z \in \mathbb{R}, q \neq 1.$$

In particular, $\forall z \in \mathbb{R}, \exp_q(\log_q(z)) = z$ and $\forall z \neq 0, \log_q(\exp_q(z)) = z$. Special case: for $q \to 1$, $\exp_q \to \exp$ and we get the exponential family.

Let us find the domain where $1 + (1-q)z > 0$:

- If $q > 1$, i.e. $1 - q < 0$ then

$$1 + (1-q)z > 0 \Leftrightarrow 1 > -(1-q)z \Leftrightarrow \frac{-1}{1-q} > z$$

- If $q < 1$, i.e. $1 - q > 0$ then

$$1 + (1-q)z > 0 \Leftrightarrow 1 > -(1-q)z \Leftrightarrow \frac{1}{1-q} < z$$

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2 q-Gaussian

Using

$$c(x) = 1/c_q,\ c_q = \sqrt{\frac{\pi}{1-q}} \frac{\Gamma(1+1/(1-q))}{\Gamma(3/2+1/(1-q))},\ H(x) = x^2,\ \alpha(\theta) = \frac{\theta^{\frac{q-1}{3-q}} - 1}{q-1},\ \theta = \sigma^{q-3},$$

we have $\alpha(\sigma) = \frac{\sigma^{q-1}-1}{q-1} = \log_{2-q}(\sigma)$. We get

$$\begin{aligned}
f_\sigma(x) &= \frac{1}{c_q} \exp_q(-\log_{2-q}(\sigma) - x^2\sigma^{q-3}) = \frac{1}{c_q} \exp_q(-\frac{\sigma^{-1+q}-1}{-1+q} - x^2\sigma^{q-3}) \\
&= \frac{1}{c_q} \exp_q(\frac{(1/\sigma)^{1-q}-1}{1-q} - x^2\sigma^{q-3}) = \frac{1}{c_q} \left[1 + (1-q)\frac{(1/\sigma)^{1-q}-1}{1-q} - (1-q)x^2\sigma^{q-3}\right]_+^{1/(1-q)} \\
&= \frac{1}{c_q} \left[(1/\sigma)^{1-q} - (1-q)x^2\sigma^{q-3}\right]_+^{1/(1-q)} = \frac{1}{c_q\sigma} \left[1 - (1-q)\sigma^{1-q}x^2\sigma^{q-3}\right]_+^{1/(1-q)} \\
&= \frac{1}{c_q\sigma} \left[1 - (1-q)x^2/\sigma^2\right]_+^{1/(1-q)} = \frac{1}{c_q\sigma} \exp_q(-x^2/\sigma^2)
\end{aligned}$$

This is different from Section 6 of Naudt (2007) where there is a typo.

3 q-Exponential

Using

$$c(x) = 1/c_q,\ c_q = \sqrt{\kappa},\ H(x) = x,\ \alpha(\theta) = \frac{\theta^{\frac{q-1}{3-q}} - 1}{q-1},\ \theta = \kappa^{\frac{q-3}{2}},$$

we get

$$f_\kappa(x) = \frac{1}{\kappa} \exp_q(-x/\kappa) = \frac{1}{\kappa} \left(1 - (1-q)\frac{x}{\kappa}\right)_+^{1/(1-q)}.$$

There exists another parametrization

$$\begin{cases} \alpha + 1 = -1/(1-q) \\ \sigma = \alpha\kappa \end{cases} \Leftrightarrow \begin{cases} \sigma = \alpha\kappa \\ -1/(\alpha+1) = 1-q \end{cases} \Leftrightarrow \begin{cases} \kappa = \sigma/\alpha \\ q = 1 + 1/(\alpha+1) \end{cases}$$

Using the parametrization (α, σ), we get the following density and distribution function

$$f(x) = \frac{\alpha}{\sigma}\left(1 + \frac{x}{\sigma}\right)_+^{-\alpha-1},\ F(x) = 1 - \left(1 + \frac{x}{\sigma}\right)_+^{-\alpha}.$$

4 Bibliography

Naudt, J. (2007), *The q-exponential family in statistical physics*, Journal of Physics: Conference Series 201 (2010) 012003.

Shalizi, C. (2007), *Maximum Likelihood Estimation for q-Exponential (Tsallis) Distributions*, http://arxiv.org/abs/math/0701854v2.